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A Note on the Spatial Decay of a Three-Dimensional Minimal Surface over a Semi-infinite Cylinder*

C. O. HORGAN

College of Engineering and Department of Mathematics, Michigan State University, East Lansing, Michigan 48224-1226

Submitted by P. D. Lax

1. INTRODUCTION

Some time ago, Knowles [1] established a spatial decay estimate for solutions $u(x_1, x_2)$ of the two-dimensional minimal surface equation

$$(1+u_{,2}^2)\, u_{,11} - 2u_{,1} u_{,2} u_{,12} + (1+u_{,1}^2)\, u_{,22} = 0 \tag{1}$$

over the semi-infinite strip $0 \leqslant x_1 < \infty$, $0 \leqslant x_2 \leqslant h$. It was shown in [1] that solutions of (1), which are continuously differentiable on the closed semi-infinite strip, twice continuously differentiable on its interior and satisfy the boundary conditions

$$u(x_1, 0) = u(x_1, h) = 0, \qquad 0 \leqslant x_1 < \infty, \tag{2}$$

$$u(x_1, x_2) \to 0 \text{ as } x_1 \to \infty, \text{ uniformly in } x_2, \qquad 0 \leqslant x_2 \leqslant h, \tag{3}$$

decay exponentially in x_1 at least as fast as do solutions of Laplace's equation subject to the same boundary conditions.

The explicit decay inequality establishing this result is given in [1] as

$$|u(x_1, x_2)| \leqslant M \sin(\pi x_2/h) \exp(-\pi x_1/h), \qquad x_1 \geqslant 0,\ 0 \leqslant x_2 \leqslant h, \tag{4}$$

where

$$M = \max_{0 \leqslant x_2 \leqslant h} \left\{ \frac{|u(0, x_2)|}{\sin(\pi x_2/h)} \right\} \geqslant 0; \tag{5}$$

M is finite since $u(0, x_2)$ is continuously differentiable for $0 \leqslant x_2 \leqslant h$ and vanishes at $x_2 = 0$ and $x_2 = h$. It is easily shown that the foregoing result

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also holds for solutions of Laplace's equation satisfying (2), (3) (see, e.g. [1, 2]); in this case the decay rate π/h is the *best possible.* Thus the presence of the nonlinear terms in (1) does not in general reduce the spatial decay rate.

The result (4) was established in [1] using arguments based on comparison (or maximum) principles for second-order nonlinear elliptic equations. Similar decay estimates were obtained for a general class of second-order quasi-linear elliptic equations in two independent variables through the use of comparison principle techniques in [3][1] and energy inequality methods in [4].

Spatial decay estimates for linear and nonlinear elliptic equations have been studied by many authors from different viewpoints and motivations. Such results have been widely used in establishing versions of Saint-Venant's principle in elasticity theory (see [2] for a recent review of this work) and in the study of spatial evolution for the stationary Navier–Stokes equations [2, 5, 6]. These results may also be interpreted as giving rise to theorems of Phragmén–Lindelöf type (see, e.g. [7, 8] for linear equations and [9, 10] and the references cited therein for nonlinear equations).

The purpose of the present note is to provide a generalization of the result (4) to solutions $u(x_1, x_2, x_3)$ in three independent variables of the three-dimensional minimal surface equation over a semi-infinite cylinder with simply-connected cross-section. It will be shown that, provided this cross-section is *convex*, an estimate analogous to (4) holds, namely that solutions of the three-dimensional minimal surface equation over such a cylinder again decay axially as least as fast as harmonic functions. Thus the foregoing behavior of solutions of the minimal surface equation is not restricted to the case of two independent variables.[2] That such a result might hold is not unexpected, and is of interest in view of the previously mentioned applications of such estimates to problems of physical relevance.

2. The Spatial Decay Estimate

Let R be the interior of a semi-infinite cylinder in three dimensions whose cross-section is bounded by a single piecewise smooth simple closed curve. Choose cartesian coordinates x_1, x_2, x_3 with the origin at the left hand end of the cylinder and the x_3-axis parallel to the generators. Let S

[1] For the special case of the minimal surface equation (1), the results in [3] are not as sharp as those of [1].

[2] Other authors have experienced difficulties in obtaining spatial decay results for nonlinear equations in three independent variables (see [9, p. 588]).

denote the open generic cross-section of R. Suppose that $u=u(x_1, x_2, x_3)$ is twice continuously differentiable on R, continuously differentiable on the closure $\bar{R}$ of R and satisfies the minimal surface equation[3]

$$\mathscr{L}u \equiv (1+u_{,i}u_{,i})\, u_{,kk} - u_{,i}u_{,j}u_{,ij} = 0 \qquad \text{on } R, \tag{6}$$

subject to the boundary conditions

$$u=0 \qquad \text{on the lateral surface } L \text{ of } R, \tag{7}$$

$$u(x_1, x_2, x_3) \to 0 \text{ as } x_3 \to \infty, \text{ uniformly in } (x_1, x_2) \text{ on } \bar{S}. \tag{8}$$

Here the usual cartesian tensor notation is employed with Latin indices ranging over 1, 2, 3, Greek indices ranging over 1, 2 and summation over repeated subscripts implied.

Let $\phi(x_1, x_2)$ be the (suitably normalized) eigenfunction corresponding to the smallest fixed membrane eigenvalue λ_1 for S, that is,

$$\phi_{,\alpha\alpha} + \lambda_1^2 \phi = 0 \qquad \text{on } S, \tag{9}$$

$$\phi = 0 \qquad \text{on the boundary } \partial S \text{ of } S. \tag{10}$$

It is well-known that we may take

$$\phi > 0 \qquad \text{on } S. \tag{11}$$

We shall establish that, *provided S is convex, for all solutions u of* (6), *subject to* (7), (8), *the inequality*

$$|u(x_1, x_2, x_3)| \leqslant K\phi(x_1, x_2) \exp(-\lambda_1 x_3), \qquad x_3 \geqslant 0, \tag{12}$$

holds on $\bar{R}$, where

$$K = \max_{\bar{S}} \{|u(x_1, x_2, 0)|/\phi(x_1, x_2)\} \geqslant 0; \tag{13}$$

K is finite by virtue of (10), (11) and the fact that $u(x_1, x_2, 0)$ is continuously differentiable on $\bar{S}$ and vanishes on ∂S. Clearly (12) is a natural generalization of the estimate (4) to three dimensions. The foregoing result also holds for harmonic functions[4] satisfying (7), (8) (see [2]) and in this case the decay rate λ_1 is the best possible.

The proof of (12) uses a comparison principle argument similar to that employed in [1] to establish (4). Thus we set

$$v(x_1, x_2, x_3) = K\phi(x_1, x_2) \exp(-\lambda_1 x_3), \tag{14}$$

[3] An equivalent version of this equation reads $(u_{,i}/W)_{,i} = 0$ on R, where $W = (1 + u_{,i}u_{,i})^{1/2}$.

[4] See also [11] for an analogous estimate for the linear heat equation. Further improvements on such estimates for the heat equation have been recently obtained in [12].

where K is given in terms of u by (13); therefore v is the solution of Laplace's equation that satisfies (7), vanishes at infinity and takes the values $K\phi(x_1, x_2)$ at $x_3 = 0$. By virtue of (11), v is nonnegative and by (13), (14) satisfies

$$-v(x_1, x_2, 0) \leqslant u(x_1, x_2, 0) \leqslant v(x_1, x_2, 0) \qquad \text{on } \bar{S}. \tag{15}$$

Also from (14), (10),

$$v = 0 \qquad \text{on the lateral surface } L. \tag{16}$$

The inequality

$$|u(x_1, x_2 x_3)| \leqslant v(x_1, x_2, x_3) \qquad \text{on } \bar{R} \tag{17}$$

now follows immediately, as in [1], from a comparison principle for second-order nonlinear elliptic equations (see, e.g. [13, 14]) once it is established that

$$\mathscr{L}v \leqslant \mathscr{L}u = 0 \qquad \text{on } R. \tag{18}$$

In view of the definition of v in (14), (17) is equivalent to the stated result (12). Thus it remains to establish conditions under which (18) holds.

A direct calculation using (14) and the differential equation (9) shows that

$$\begin{aligned} \mathscr{L}v = & -K^3 \exp(-3\lambda_1 x_3)[\lambda_1^2 \phi\phi_{,\alpha}\phi_{,\alpha} + \phi_{,\alpha}\phi_{,\beta}\phi_{,\alpha\beta} \\ & + \lambda_1^2 \phi(\lambda_1^2 \phi^2 + \phi_{,\alpha}\phi_{,\alpha})] \\ \equiv & -K^3 \exp(-3\lambda_1 x_3)\, Q(\phi) \end{aligned} \tag{19}$$

and so (18) will follow provided

$$Q(\phi) \geqslant 0 \qquad \text{on } S. \tag{20}$$

Now the curvature k of the level curve $\phi = \text{constant}$ is given by (see [15])

$$kq^3 = \phi_{,\alpha}\phi_{,\beta}\phi_{,\alpha\beta} - q^2\phi_{,\alpha\alpha} = \phi_{,\alpha}\phi_{,\beta}\phi_{,\alpha\beta} + \lambda_1^2 q^2 \phi \qquad (q = |\text{grad } \phi|), \tag{21}$$

and so

$$Q(\phi) = kq^3 + \lambda_1^2 \phi(\lambda_1^2 \phi^2 + q^2) \qquad \text{on } S. \tag{22}$$

It has been established recently in [15] that

$$k \geqslant 0 \qquad \text{on } S \text{ for convex domains } S, \tag{23}$$

and so, for convex S, it follows from (22), (23) that (20) holds. This completes the proof of (12).

3. Generalizations

The result (12) can, in fact, be established for a wide class of quasilinear elliptic equations.[5] Thus suppose that, instead of the minimal surface equation (6), we consider the elliptic equation

$$Pu \equiv [\rho(q^2)\, u_{,j}]_{,j} = 0 \qquad \text{on } R, \tag{24}$$

where $q^2 = |\nabla u|^2$. The argument of Section 2 leading to the conclusion (12) still holds provided we can show that $Pv \leqslant 0$ on R, where v is given by (14). A direct calculation using (14) and (9) shows that

$$Pv = K^3 \rho' \exp(-3\lambda_1 x_3)\, Q(\phi), \tag{25}$$

where $Q(\phi)$ has been defined in (19), and $\rho' = \partial\rho/\partial q^2$ (evaluated at v). Since $Q(\phi) \geqslant 0$ for convex domains S, it follows from (25) that $Pv \leqslant 0$ on R if

$$\rho' \leqslant 0 \qquad \text{on } R. \tag{26}$$

Thus for solutions of quasilinear elliptic equations of the form (24), satisfying (7), (8), the result (12) holds provided S is *convex* and condition (26) is satisfied.[6] It is easily shown that for even more general elliptic equations (24), with $\rho = \rho(u, q^2)$, the same result holds if, in addition to (26), we have $\partial\rho/\partial u \leqslant 0$ on R. Decay estimates for such equations, with different hypotheses on ρ, have been recently obtained in [16] using energy inequality techniques.

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[5] The author is grateful to L. E. Payne for drawing this to his attention.

[6] A similar argument can be applied to obtain the result (4) for *two-dimensional* equations of the form (24) on a semi-infinite strip, with $\rho' \leqslant 0$, thus generalizing the result of [1] to such equations.

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